CONTROLLABILITY OF NEUTRAL STOCHASTIC FUNCTIONAL INTEGRO-DIFFERENTIAL EQUATIONS DRIVEN BY FRACTIONAL BROWNIAN MOTION

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Abstract. This paper focuses on controllability results of stochastic delay partial functional integro-differential equations perturbed by fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$. Sufficient conditions are established using the theory of resolvent operators developed by R. Grimmer in [8] combined with a fixed point approach for achieving the required result. An example is provided to illustrate the theory.

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1. Introduction

The noise or perturbations of a system are typically modeled by a Brownian motion as such a process is Gauss-Markov and has independent increments. However, empirical data from many physical phenomena suggest that Brownian motion is often shown not to be an effective process to use in a model. A family of processes that seems to have wide physical applicability is fractional Brownian motion (fBm). This process was introduced by Kolmogorov in [10] and later studied by Mandelbrot and Van Ness in [12], where a stochastic integral representation in term of a standard Brownian motion was obtained. Since the fBm B^H is not a semimartingale if $H \neq \frac{1}{2}$ (see [1]), we can not use the classical Itô theory to construct a stochastic calculus with respect to fBm.

Since some physical phenomena are naturally modeled by stochastic partial differential equations or stochastic integro-differential equations and the randomness can be described by a fBm, it is important to study the controllability of infinite dimensional equations with a fBm. Many studies of the solutions of stochastic equations in an infinite dimensional space with a fBm have been emerged recently, see [1, 2, 3, 4, 6, 11, 13]. The literature related to neutral stochastic partial functional integro-differential equations driven by a fBm is not vast. Very recently, in [5], the authors studied the existence and uniqueness of mild solutions for a class of

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stochastic delay partial functional integro-differential equations by using the theory of resolvent operators.

The control problems for stochastic equations driven by fractional noise have been studied only recently and no results seem to be available for controllability. Motivated by [5, 17], but the analysis for fBm requires additional results and we have to construct a new control, moreover, we study the controllability for the following neutral stochastic delay partial functional integro-differential equations perturbed by a fractional Brownian motion:

$$\begin{cases} d[x(t)+G(t,x(t-r(t)))] & = [Ax(t)+G(t,x(t-r(t)))+Hu(t)]dt \\ & + \int_0^t B(t-s)[x(s)+G(s,x(s-r(s)))]dsdt \\ & +F(t,x(t-\rho(t)))dt+\sigma(t)dB^H(t),\ 0\leq t\leq T, \\ x(t)=\varphi(t),\ -\tau\leq t\leq 0. & \end{cases} \tag{1.1}$$

Here A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators, $(S(t))_{t\geq 0}$, in a Hilbert space X with domain $D(A)$, $B(t)$ is a closed linear operator on X with domain $D(B(t))\supset D(A)$ which independent of t. The control function $u(.)$ takes values in $L^2([0,T],U)$, the Hilbert space of admissible control functions for a separable Hilbert space U. The symbol H stands for a bounded linear operator from U into X. B^H is a Fractional Brownian motion on a real and separable Hilbert space Y, $r,\ \rho:[0,+\infty)\to[0,\tau]$, $(\tau>0)$ are continuous and $F,G:[0,+\infty)\times X\to X$, $\sigma:[0,+\infty)\to\mathcal{L}_2^0(Y,X)$ are appropriate functions. Here $\mathcal{L}_2^0(Y,X)$ denotes the space of all Q-Hilbert-Schmidt operators from Y into X (see section 2 below).

In this paper, we study the controllability result with the help of resolvent operators. The resolvent operator is similar to the evolution operator for nonautonomous differential equations in a Hilbert spaces. It will not, however, be an evolution operator because it will not satisfy an evolution or semigroup property. On the other hand, to the best of our knowledge, there is no paper which investigates the controllability of neutral stochastic integro-differential equations with delays driven by a fractional Brownian motion . Thus, we will make the first attempt to study such problem in this paper.

The rest of this paper is organized as follows, In Section 2, we introduce some notations, concepts, and basic results about fractional Brownian motion, Wiener integral over Hilbert spaces and we mention a few results and notations related with resolvent of operators. In Section 3, the controllability of the system (1.1) is investigated via a fixed-point analysis approach. Example presented in Section 4 demonstrates the controllability result of section 3.

2. Preliminaries

In this section, we recall some fundamental results needed to establish our results. For details of this section, we refer the reader to [14, 8] and references therein.

2.1. **Fractional Brownian motion.** Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space satisfying the usual condition, which means that the filtration is right continuous increasing family and $\mathcal{F}_0$ contains all P-null sets.

Consider a time interval $[0, T]$ with arbitrary fixed horizon T and let $\{\beta^H(t), t \in [0, T]\}$ the one-dimensional fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$. This means by definition that β^H is a centered Gaussian process with covariance function:

$$R_H(s,t) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

Moreover β^H has the following Wiener integral representation:

$$\beta^H(t) = \int_0^t K_H(t,s)d\beta(s), \tag{2.1}$$

where $\beta = \{\beta(t) : \ t \in [0, T]\}$ is a Wiener process, and $K_H(t; s)$ is the kernel given by

$$K_H(t,s) = c_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du,$$

for $t > s$, where $c_H = \sqrt{\frac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2})}}$ and $\beta(,)$ denotes the Beta function. We put $K_H(t, s) = 0$ if $t \leq s$.
We will denote by $\mathcal{H}$ the reproducing kernel Hilbert space of the fBm. In fact $\mathcal{H}$ is the closure of set of indicator functions $\{1_{[0;t]}, t \in [0, T]\}$ with respect to the scalar product

$$\langle 1_{[0,t]}, 1_{[0,s]} \rangle_{\mathcal{H}} = R_H(t, s).$$

The mapping $1_{[0,t]} \to \beta^H(t)$ can be extended to an isometry between $\mathcal{H}$ and the first Wiener chaos and we will denote by $\beta^H(\varphi)$ the image of φ by the previous isometry.

We recall that for $\psi, \varphi \in \mathcal{H}$ their scalar product in $\mathcal{H}$ is given by

$$\langle \psi, \varphi \rangle_{\mathcal{H}} = H(2H-1) \int_0^T \int_0^T \psi(s)\varphi(t)|t-s|^{2H-2} dsdt.$$

Let us consider the operator K_H^* from $\mathcal{H}$ to $L^2([0, T])$ defined by

$$(K_H^* \varphi)(s) = \int_s^T \varphi(r) \frac{\partial K}{\partial r}(r, s) dr.$$

We refer to [14] for the proof of the fact that K_H^* is an isometry between $\mathcal{H}$ and $L^2([0, T])$. Moreover for any $\varphi \in \mathcal{H}$, we have

$$\beta^H(\varphi) = \int_0^T (K_H^* \varphi)(t) d\beta(t).$$

It follows from [14] that the elements of $\mathcal{H}$ may be not functions but distributions of negative order. In order to obtain a space of functions contained in $\mathcal{H}$, we consider the linear space $|\mathcal{H}|$ generated by the measurable functions ψ such that

$$\|\psi\|^2_{|\mathcal{H}|} := \alpha_H \int_0^T \int_0^T |\psi(s)||\psi(t)||s-t|^{2H-2} dsdt < \infty,$$

where $\alpha_H = H(2H-1)$. We have the following Lemma (see [14])

Lemma 2.1. *The space $|\mathcal{H}|$ is a Banach space with the norm $\|\psi\|_{|\mathcal{H}|}$ and we have the following inclusions*

$$\mathbb{L}^2([0,T]) \subseteq \mathbb{L}^{1/H}([0,T]) \subseteq |\mathcal{H}| \subseteq \mathcal{H},$$

and for any $\varphi \in \mathbb{L}^2([0,T])$, we have

$$\|\psi\|^2_{|\mathcal{H}|} \leq 2HT^{2H-1} \int_0^T |\psi(s)|^2 ds.$$

Let X and Y be two real, separable Hilbert spaces and let $\mathcal{L}(Y,X)$ be the space of bounded linear operator from Y to X. For the sake of convenience, we shall use the same notation to denote the norms in X, Y and $\mathcal{L}(Y,X)$. Let $Q \in \mathcal{L}(Y,Y)$ be an operator defined by $Qe_n = \lambda_n e_n$ with finite trace $trQ = \sum_{n=1}^{\infty} \lambda_n < \infty$. where $\lambda_n \geq 0$ $(n = 1, 2...)$ are non-negative real numbers and $\{e_n\}$ $(n = 1, 2...)$ is a complete orthonormal basis in Y. Let $B^H = (B^H(t))$ be $Y-$ valued fbm on $(\Omega, \mathcal{F}, \mathbb{P})$ with covariance Q as

$$B^H(t) = B^H_Q(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} e_n \beta^H_n(t),$$

where β^H_n are real, independent fBm's. This process is Gaussian, it starts from 0, has zero mean and covariance:

$$E\langle B^H(t), x\rangle \langle B^H(s), y\rangle = R(s,t) \langle Q(x), y\rangle \text{ for all } x, y \in Y \text{ and } t, s \in [0,T].$$

In order to define Wiener integrals with respect to the Q-fBm, we introduce the space $\mathcal{L}^0_2 := \mathcal{L}^0_2(Y,X)$ of all Q-Hilbert-Schmidt operators $\psi : Y \to X$. We recall that $\psi \in \mathcal{L}(Y,X)$ is called a Q-Hilbert-Schmidt operator, if

$$\|\psi\|^2_{\mathcal{L}^0_2} := \sum_{n=1}^{\infty} \|\sqrt{\lambda_n} \psi e_n\|^2 < \infty,$$

and that the space $\mathcal{L}^0_2$ equipped with the inner product $\langle \varphi, \psi\rangle_{\mathcal{L}^0_2} = \sum_{n=1}^{\infty} \langle \varphi e_n, \psi e_n\rangle$ is a separable Hilbert space.

Now, let $\phi(s)$; $s \in [0,T]$ be a function with values in $\mathcal{L}^0_2(Y,X)$, The Wiener integral of ϕ with respect to B^H is defined by

$$\int_0^t \phi(s) dB^H(s) = \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda_n} \phi(s) e_n d\beta^H_n(s) = \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda_n} (K^*_H(\phi e_n)(s) d\beta_n(s), \tag{2.2}$$

where β_n is the standard Brownian motion used to present β^H_n as in (2.1).
Now, we end this subsection by stating the following result which is fundamental to prove our result. It can be proved by similar arguments as those used to prove Lemma 2 in [4].

Lemma 2.2. *If $\psi : [0,T] \to \mathcal{L}^0_2(Y,X)$ satisfies $\int_0^T \|\psi(s)\|^2_{\mathcal{L}^0_2} ds < \infty$ then the above sum in* (2.2) *is well defined as a X-valued random variable and we have*

$$\mathbb{E}\| \int_0^t \psi(s) dB^H(s)\|^2 \leq 2Ht^{2H-1} \int_0^t \|\psi(s)\|^2_{\mathcal{L}^0_2} ds.$$

2.2. **Partial Integro-differential Equations.** For better comprehension of the subject we shall introduce some definitions, hypothesis and results. We refer the reader to [8]. Throughout the rest of the paper we always assume that X is a Banach space, A and $B(t)$ are closed linear operators on X. Y represents the Banach space $D(A)$ equipped with the graph norm defined by

$$\|y\|_Y = \|Ay\| + \|y\|, \qquad \text{for } y \in Y.$$

We consider the following the abstract integro-differential problem

$$\frac{dx(t)}{dt} = Ax(t) + \int_0^t B(t-s)x(s)\,ds, \tag{2.3}$$

$$x(0) = x \in X. \tag{2.4}$$

Definition 2.3. A one-parameter family of bounded linear operators $(R(t))_{t\geq 0}$ on X is called a resolvent operator of (2.3)-(2.4) if the following conditions are satisfied.

(a) $R(\cdot) : [0,\infty) \to \mathcal{L}(X)$ is strongly continuous and $R(0)x = x$ for all $x \in X$.
(b) For $x \in D(A)$, $R(\cdot)x \in C([0,\infty),[D(A)]) \cap C^1([0,\infty),X)$, and

$$\frac{dR(t)x}{dt} = AR(t)x + \int_0^t B(t-s)R(s)xds, \tag{2.5}$$

$$\frac{dR(t)x}{dt} = R(t)Ax + \int_0^t R(t-s)B(s)xds, \tag{2.6}$$

for every $t \geq 0$,
(c) There exists some constants $M > 0, \delta$ such that $\|R(t)\| \leq Me^{\delta t}$ for every $t \geq 0$.

Definition 2.4. A resolvent operator $(R(t))_{t\geq 0}$ of (2.3)-(2.4) is called exponentially stable if there exists positive constants M, α such that $\|R(t)\| \leq Me^{-\alpha t}$.

The resolvent operators play an important role to study the existence of solutions and to give a variation of constants formula for nonlinear systems. We need to know when the linear system (2.3)-(2.4) has a resolvent operator. For more details on resolvent operators, we refer to [8, 7]. In this work we assume that the following conditions are satisfied:

($\mathcal{A}$.1) A is the infinitesimal generator of a strongly continuous semigroup on X.
($\mathcal{A}$.2) For all $t \geq 0$, $B(t)$ is a closed linear operator from $D(A)$ to X, and $B(t) \in \mathcal{L}(Y,X)$. For any $y \in Y$, the map $t \longrightarrow B(t)y$ is bounded, differentiable and the derivative $B'(t)y$ is bounded and uniformly continuous on $\mathbb{R}^+$.

Theorem 2.5. [8, Theorem 3.7] *Assume that* ($\mathcal{A}$.1) *and* ($\mathcal{A}$.2) *hold. Then there exists a unique resolvent operator of the Cauchy problem (2.3)-(2.4).*

In the remaining of this section we discuss the existence of solutions to

$$\frac{dx(t)}{dt} = Ax(t) + \int_0^t B(t-s)x(s)\;ds + f(t), \quad t \in\geq 0, \tag{2.7}$$

$$x(0) = z \in X, \tag{2.8}$$

where $f : [0,+\infty) \longrightarrow X$ is a continuous function. We begin by introducing the following concept of strict solution.

Definition 2.6. *A function $x : [0,+\infty) \to X$, is called a strict solution of* (2.7)-(2.8) *on* $[0,+\infty)$ *if* $x \in C([0,+\infty),[D(A)]) \cap C^1([0,+\infty),X)$*, the condition* (2.8) *holds and the equation* (2.7) *is satisfied on* $[0,+\infty)$.

Theorem 2.7 ([8, Theorem 2.5])**.** *Let $z \in X$. Assume that $f \in C([0,+\infty),X)$ and $x(\cdot)$ is a strict solution of* (2.7)-(2.8) *on* $[0,+\infty)$*. Then*

$$x(t) = R(t)z + \int_0^t R(t-s)f(s)\,ds, \quad t \in [0,+\infty). \tag{2.9}$$

Motivated by (2.9), we introduce the following concept.

Definition 2.8. A function $u \in C([0,+\infty),X)$ is called a mild solution of (2.7)-(2.8) if

$$u(t) = R(t)z + \int_0^t R(t-s)f(s)\,ds, \quad t \in [0,T], \text{ for } z \in X.$$

3. Controllability Result

In this section we study the controllability results for Equation (1.1). Before starting, we introduce the concept of a mild solution of the problem (1.1) and controllability of neutral integro-differential stochastic functional differential equation. Motivated by the theory of resolvent operator, we introduce the following concept of mild solution for equation (1.1).

Definition 3.1. *An X-valued stochastic process $\{x(t),\ t \in [-\tau,T]\}$, is called a mild solution of equation (1.1) if*

i) $x(.) \in \mathcal{C}([-\tau,T],\mathbb{L}^2(\Omega,X))$,

ii) $x(t) = \varphi(t)$, $-\tau \le t \le 0$.

iii) For arbitrary $t \in [0,T]$, we have

$$\begin{array}{rcl} x(t) & = & R(t)(\varphi(0) + G(0,\varphi(-r(0)))) - G(t,x(t-r(t))) \\ & + & \int_0^t R(t-s)[Hu(s) + F(s,x(s-\rho(s)))]ds \\ & + & \int_0^t R(t-s)\sigma(s)dB^H(s) \quad \mathbb{P}-a.s. \end{array} \tag{3.1}$$

Definition 3.2. *The system (1.1) is said to be controllable on the interval $[-\tau,T]$, if for every initial stochastic process φ defined $[-\tau,0]$ and $x_1 \in X$, there exists a stochastic control $u \in L^2([0,T],U)$ such that the mild solution $x(.)$ of (1.1) satisfies $x(T) = x_1$.*

Roughly speaking, controllability problem for evolution system consists in driving the state of the system (the mild solution of the controlled equation under consideration) from an arbitrary initial state to an arbitrary final state in finite time.

To prove the controllability result, we consider the following assumptions:

($\mathcal{H}.1$) The resolvent operator $(R(t))_{t\ge 0}$ given by ($\mathcal{A}.1$) ($\mathcal{A}.2$) satisfies the following condition: there is a positive constant M such that

$$\sup_{0\le s,t\le T} \|R(t-s)\| \le M.$$

($\mathcal{H}.2$) The function $f : [0,+\infty) \times X \to X$ satisfies the following Lipschitz conditions: that is, there exist positive constants $C_i := C_i(T), i = 1,2$ such that, for all $t \in [0,T]$ and $x,y \in X$

(i) $\|F(t,x) - F(t,y)\| \le C_1\|x-y\|$.

(ii) $\|F(t,x)\|^2 \leq C_2(1+\|x\|^2)$.

$(\mathcal{H}.3)$ The function $G : [0,+\infty)\times X \longrightarrow X$ satisfies the following conditions: there exist positive constants C_3 and C_4, $C_3 < \frac{1}{2}$, such that, for all $t \in [0,T]$ and $x, y \in X$

(i) $\|G(t,x) - G(t,y)\| \leq C_3\|x-y\|$.

(ii) $\|G(t,x)\|^2 \leq C_4(1+\|x\|^2)$.

$(\mathcal{H}.4)$ The function G is continuous in the quadratic mean sense:

$$\text{For all } \; x \in \mathcal{C}([0,T], \mathbb{L}^2(\Omega, X)), \;\; \lim_{t\to s} \mathbb{E}\|G(t,x(t)) - G(s,x(s))\|^2 = 0.$$

$(\mathcal{H}.5)$ The function $\sigma : [0,+\infty) \to \mathcal{L}_2^0(Y,X)$ satisfies

$$\int_0^T \|\sigma(s)\|^2_{\mathcal{L}_2^0} ds < \infty, \;\; \forall T > 0.$$

$(\mathcal{H}.6)$ The linear operator W from U into X defined by

$$Wu = \int_0^T R(T-s)Hu(s)ds$$

has an inverse operator W^{-1} that takes values in $L^2([0,T],U) \setminus kerW$, where $kerW = \{x \in L^2([0,T],U), \;\; Wx = 0\}$ (see [9]), and there exists finite positive constants M_b, M_w such that $\|B\| \leq M_b$ and $\|W^{-1}\| \leq M_w$.

Moreover, we assume that $\varphi \in \mathcal{C}([-\tau,0], \mathbb{L}^2(\Omega,X))$.

We can now state the main result of this paper.

Theorem 3.3. *Suppose that $(\mathcal{H}.1) - (\mathcal{H}.6)$ hold. Then, the system (1.1) is controllable on $[-\tau,T]$.*

Proof. Fix $T > 0$ and let $\mathcal{B}_T := \mathcal{C}([-\tau,T], \mathbb{L}^2(\Omega,X))$ be the Banach space of all continuous functions from $[-\tau,T]$ into $\mathbb{L}^2(\Omega,X)$, equipped with the supremum norm $\|\xi\|_{\mathcal{B}_T} = \sup_{u\in[-\tau,T]} \left(\mathbb{E}\|\xi(u)\|^2\right)^{1/2}$ and let us consider the set

$$S_T = \{x \in \mathcal{B}_T : x(s) = \varphi(s), \text{ for } \; s \in [-\tau,0]\}.$$

S_T is a closed subset of $\mathcal{B}_T$ provided with the norm $\|.\|_{\mathcal{B}_T}$.

Using the hypothesis (H6) for an arbitrary function $x(.)$, define the control

$$\begin{aligned} u(t) &= W^{-1}\{x_1 - R(T)(\varphi(0) + G(0,\varphi(-r(0)))) + G(T, x(T - r(T))) \\ &- \int_0^T R(T-s)F(s-\rho(s))ds - \int_0^T R(T-s)\sigma(s)dB^H(s)\}(t). \end{aligned} \tag{3.2}$$

We shall now show that when using this control, the operator Φ defined on S_T by $\Phi(x)(t) = \varphi(t)$ for $t \in [-\tau,0]$ and for $t \in [0,T]$

$$\begin{aligned} \Phi(x)(t) &= R(t)(\varphi(0) + G(0,\varphi(-r(0)))) - G(t, x(t-r(t))) \\ &+ \int_0^t R(t-s)[Hu(s) + F(s-\rho(s))]ds] + \int_0^t R(t-s)\sigma(s)dB^H(s) \end{aligned} \tag{3.3}$$

has a fixed point. Substituting (3.2) in (3.3) we can show that $\psi x(T) = x_1$, which means that the control u steers the system from the initial state φ to x_1 in time T, provided we can obtain a fixed point of the operator ψ which implies that the system in controllable.

Next we will show by using Banach fixed point theorem that ψ has a unique fixed point. We divide the subsequent proof into two steps.

Step 1: For arbitrary $x \in S_T$, let us prove that $t \to \Phi(x)(t)$ is continuous on the interval $[0,T]$ in the $\mathbb{L}^2(\Omega, X)$-sense.

Let $0 < t < T$ and $|h|$ be sufficiently small. Then for any fixed $x \in S_T$, we have

$$\begin{aligned}
\mathbb{E}\|\Phi(x)(t & + & h) - \Phi(x)(t)\|^2 \leq 5\mathbb{E}\|(R(t+h) - R(t))(\varphi(0) + G(0, \varphi(-r(0))))\| \\
& + & 5\mathbb{E}\|G(t+h, x(t+h-r(t+h))) - G(t, x(t-r(t)))\| \\
& + & 5\mathbb{E}\|\int_0^{t+h} R(t+h-s)F(s, x(s-\rho(s)))ds - \int_0^t R(t-s)F(s, x(s-\rho(s)))ds\|^2 \\
& + & 5\mathbb{E}\|\int_0^{t+h} R(t+h-s)\sigma(s)dB^H(s) - \int_0^t R(t-s)\sigma(s)dB^H(s)\| \\
& + & 5\mathbb{E}\|\int_0^{t+h} R(t+h-\nu)HW^{-1}\{x_1 - R(T)(\varphi(0) + G(0, \varphi(-r(0)))) \\
& + & G(T, x(T-r(T))) - \int_0^T R(T-s)F(s, x(s-\rho(s)))ds \\
& - & \int_0^T R(T-s)\sigma(s)dB^H(s)\}d\nu \\
& - & \int_0^t R(t-\nu)HW^{-1}\{x_1 - R(T)(\varphi(0) + G(0, \varphi(-r(0)))) + G(T, x(T-r(T))) \\
& - & \int_0^T R(T-s)F(s, x(s-\rho(s)))ds - \int_0^T R(T-s)\sigma(s)dB^H(s)\}d\nu \\
& = & \sum_{1 \leq i \leq 5} 5\mathbb{E}\|I_i(h)\|^2.
\end{aligned}$$

We are going to show that each function $t \to I_i(t)$ is continuous on $[0,T]$ in the L^2 sens.

By the strong continuity of $R(t)$, we have

$$\lim_{h \to 0} (R(t+h) - R(t))(\varphi(0) + G(0, \varphi(-r(0)))) = 0.$$

The condition $(\mathcal{H}.1)$ assures that

$$\|(R(t+h) - R(t))(\varphi(0) + G(0, \varphi(-r(0))))\| \leq 2M\|\varphi(0) + G(0, \varphi(-r(0)))\| \in \mathbb{L}^2(\Omega).$$

Then we conclude by the Lebesgue dominated theorem that

$$\lim_{h \to 0} \mathbb{E}\|I_1(h)\|^2 = 0.$$

By using the fact that the operator G is continuous in the quadratic mean sense, we conclude by condition $(\mathcal{H}.4)$ that

$$\lim_{h \to 0} \mathbb{E}\|I_2(h)\|^2 = 0.$$

For the third term $I_3(h)$, we suppose that $h > 0$ (Similar estimates hold for $h < 0$), then we have

$$\|I_3(h)\| \quad \leq \quad \|\int_0^t (R(t+h-s) - R(t-s))F(s, x(s-r(s)))ds\|$$

$$\begin{aligned} &+\|\int_t^{t+h} R(t+h-s)F(s,x(s-r(s)))ds\| \\ \leq\ & I_{31}(h)+I_{32}(h). \end{aligned}$$

By Hölder's inequality, one has that

$$\mathbb{E}|I_{31}(h)|^2 \leq t\mathbb{E}\int_0^t \|(R(t+h-s)-R(t-s))F(s,x(s-r(s)))\|^2 ds$$

By using the strong continuity of $R(t)$, we have for each $s \in [0,t]$,

$$\lim_{h\to 0}(R(t+h-s)-R(t-s))F(s,x(s-r(s))) = 0.$$

By using condition $(\mathcal{H}.1)$, condition (ii) in $(\mathcal{H}.2)$, we obtain

$$\begin{aligned} &\|(R(t+h-s)-R(t-s))F(s,x(s-r(s)))\|^2 \\ &\leq 4M^2\|F(s,x(s-r(s)))\|^2, \end{aligned}$$

then, we conclude by the Lebesgue dominated theorem that

$$\lim_{h\to 0}\mathbb{E}\|I_{31}(h)\|^2 = 0.$$

By conditions $(\mathcal{H}.1)$, $(\mathcal{H}.2)$ and Hölder's inequality, we get

$$\mathbb{E}\|I_{32}(h)\|^2 \leq C_2^2 hM^2\int_0^T (\mathbb{E}\|x(s-r(s))\|^2+1)ds,$$

then

$$\lim_{h\to 0}\mathbb{E}\|I_3(h)\|^2 = 0.$$

For the term $I_4(h)$, we have

$$\begin{aligned} \|I_4(h)\| \quad &\leq \quad \|\int_0^t (R(t+h-s)-R(t-s))\sigma(s)dB^H(s)\| \\ &+ \quad \|\int_t^{t+h} R(t+h-s)\sigma(s)dB^H(s)\| \\ &\leq \quad I_{41}(h)+I_{42}(h). \end{aligned}$$

By condition $(\mathcal{H}.1)$ and Lemma 2.2, we get that

$$E|I_{41}(h)|^2 \quad \leq \quad 2Ht^{2H-1}\int_0^t \|(R(t+h-s)-R(t-s))\sigma(s)\|^2_{\mathcal{L}_2^0} ds$$

Since $\lim_{h\to 0}\|(R(t+h-s)-R(t-s))\sigma(s)\|^2_{\mathcal{L}_2^0} = 0$ and

$$\|(R(t+h-s)-R(t-s))\sigma(s)\|^2_{\mathcal{L}_2^0} \leq 4M^2\|\sigma(s)\|^2_{\mathcal{L}_2^0} \in \mathbb{L}^1([0,T],ds),$$

we conclude, by the dominated convergence theorem that,

$$\lim_{h\to 0}\mathbb{E}|I_{41}(h)|^2 = 0.$$

Again by Lemma 2.2, we get that

$$\mathbb{E}|I_{42}(h)|^2 \leq 2Hh^{2H-1}M^2\int_t^{t+h} \|\sigma(s)\|^2_{\mathcal{L}_2^0} ds \to 0.$$

Next, let's observe that

$$\begin{aligned}
\mathbb{E}\|I_5(h)\|^2 &\leq 2\mathbb{E}\|\int_t^{t+h} R(t+h-\nu)BW^{-1}\{x_1 - R(T)(\varphi(0)+G(0,\varphi(-r(0))))\\
&+G(T,x(T-r(T))) - \int_0^T R(T-s)F(s,x(s-\rho(s)))ds\\
&-\int_0^T R(T-s)\sigma(s)dB^H(s)\}d\nu\|^2\\
&+2\mathbb{E}\|\int_0^t (R(t+h-\nu)-R(t-\nu))BW^{-1}\{x_1 - R(T)(\varphi(0)+G(0,\varphi(-r(0))))\\
&+G(T,x(T-r(T))) - \int_0^T R(T-s)F(s,x(s-\rho(s)))ds\\
&-\int_0^T R(T-s)\sigma(s)dB^H(s)\}d\nu\|^2\\
&\leq 2[\mathbb{E}\|I_{5,1}(h)\|^2 + \mathbb{E}\|I_{5,2}(h)\|^2].
\end{aligned}$$

Let's first deal with $I_{5,1}(h)$, using conditions $(\mathcal{H}.1)-(\mathcal{H}.6)$ and Hölder inequality, it follows that

$$\begin{aligned}
\mathbb{E}\|I_{5,1}(h)\|^2 &\leq 5M^2M_b^2M_w^2\int_t^{t+h}\{\mathbb{E}\|x_1\|^2 + M^2\mathbb{E}\|\varphi(0)+G(0,\varphi(-r(0)))\|^2\\
&+C_4^2(1+\sup_{s\in[-\tau,T]}\mathbb{E}\|x(s)\|^2) + M^2TC_2^2(1+\sup_{s\in[-\tau,T]}\mathbb{E}\|x(s)\|^2)\\
&+2M^2HT^{2H-1}\int_0^T\|\sigma(s)\|^2_{\mathcal{L}_2^0}ds\}d\nu.
\end{aligned}$$

It results that

$$\lim_{h\to 0}\mathbb{E}||I_{5,1}(h)||^2 = 0.$$

In a similar way, we have

$$\begin{aligned}
\mathbb{E}\|I_{5,2}(h)\|^2 &\leq 5M_b^2M_w^2\int_0^t\|(R(t+h-\nu)-R(t-\nu))\|^2\{\mathbb{E}\|x_1\|^2\\
&+M^2\mathbb{E}\|\varphi(0)+G(0,\varphi(-r(0)))\|^2 + C_4^2(1+\sup_{s\in[-\tau,T]}\mathbb{E}\|x(s)\|^2)\\
&+M^2T^2C_2^2(1+\sup_{s\in[-\tau,T]}\mathbb{E}\|x(s)\|^2) + 2M^2HT^{2H-1}\int_0^T\|\sigma(s)\|^2_{\mathcal{L}_2^0}ds\}d\nu.
\end{aligned}$$

Since

$$\begin{aligned}
&\|R(t+h-\nu)-R(t-\nu)\|^2\{\mathbb{E}\|x_1\|^2 + M^2\mathbb{E}\|\varphi(0)+G(0,\varphi(-r(0)))\|^2\\
&+C_4^2(1+\sup_{s\in[-\tau,T]}\mathbb{E}\|x(s)\|^2) + M^2T^2C_2^2(1+\sup_{s\in[-\tau,T]}\mathbb{E}\|x(s)\|^2)\\
&+2M^2HT^{2H-1}\int_0^T\|\sigma(s)\|^2_{\mathcal{L}_2^0}ds\}\\
&\leq 4M^2\{\mathbb{E}\|x_1\|^2 + M^2\mathbb{E}\|\varphi(0)+G(0,\varphi(-r(0)))\|^2 + C_4^2(1+\sup_{s\in[-\tau,T]}\mathbb{E}\|x(s)\|^2)\\
&+M^2T^2C_2^2(1+\sup_{s\in[-\tau,T]}\mathbb{E}\|x(s)\|^2) + 2M^2HT^{2H-1}\int_0^T\|\sigma(s)\|^2_{\mathcal{L}_2^0}ds\} \in L^1([0,T],ds]),
\end{aligned}$$

we conclude, by the dominated convergence theorem that,

$$\lim_{h\to 0}\mathbb{E}||I_{5,2}(h)||^2 = 0.$$

The above arguments show that $\lim_{h\to 0}\mathbb{E}\|\Phi(x)(t+h)-\Phi(x)(t)\|^2=0$. Hence, we conclude that the function $t\to\Phi(x)(t)$ is continuous on $[0,T]$ in the $\mathbb{L}^2$-sense.

Step 2. Now, we are going to show that Φ is a contraction mapping in S_{T_1} with some $T_1\leq T$ to be specified later.

Let $x,y\in S_T$ we obtain for any fixed $t\in[0,T]$

$$\begin{aligned}
\|\Phi(x)(t) \quad - \quad & \Phi(y)(t)\|^2\\
\leq \quad & 4\|G(t,x(t-r(t)))-G(t,y(t-r(t)))\|^2\\
& +4\|\int_0^t R(t-s)[F(s,x(s-\rho(s)))-F(s,y(s-\rho(s)))]ds\|^2\\
& +4\|\int_0^t R(t-\nu)BW^{-1}[G(T,x(T-r(T)))-G(T,y(T-r(T)))]d\nu\|^2\\
& +4\|\int_0^t R(t-\nu)BW^{-1}\int_0^T R(T-s)[F(s,x(s-\rho(s)))-F(s,y(s-\rho(s)))]dsd\nu\|^2.
\end{aligned}$$

By Lipschitz property of F and G combined with Hölder's inequality, we obtain

$$\begin{aligned}
\mathbb{E}\|\Phi(x)(t)-\Phi(y)(t)\|^2 \quad \leq \quad & 4C_3^2\mathbb{E}\|x(t-r(t))-y(t-r(t))\|^2\\
& +4M^2C_1^2t\int_0^t\mathbb{E}\|x(s-r(s))-y(s-r(s))\|^2ds\\
& +4\mathbf{t}M^2M_b^2M_w^2[\mathbb{E}\|x(T-r(T))-y(T-r(T))\|^2\\
& +T^2C_1^2M^2\sup_{s\in[-\tau,t]}\mathbb{E}\|x(s)-y(s)\|^2.
\end{aligned}$$

Hence

$$\sup_{s\in[-\tau,t]}\mathbb{E}\|\Phi(x)(s)-\Phi(y)(s)\|^2\leq\gamma(t)\sup_{s\in[-\tau,t]}\mathbb{E}\|x(s)-y(s)\|^2.$$

where

$$\gamma(t)=4[C_3^2+M^2C_1^2t^2+\mathbf{t}M^2M_b^2M_w^2(1+T^2C_1^2M^2)].$$

By condition (iii) in $(\mathcal{H}.3)$, we have $\gamma(0)=4C_3^2<1$. Then there exists $0<T_1\leq T$ such that $0<\gamma(T_1)<1$ and Φ is a contraction mapping on S_{T_1} and therefore has a unique fixed point, which is a mild solution of equation (1.1) on $[-\tau,T_1]$. This procedure can be repeated in order to extend the solution to the entire interval $[-\tau,T]$ in finitely many steps. Clearly, $(\psi x)(T)=x_1$ which implies that the system (1.1) is controllable on $[-\tau,T]$. This completes the proof. □

4. Example

We consider the following stochastic partial neutral functional integro-differential equation with finite delays τ_1 and τ_2 $(0\leq\tau_i\leq\tau<\infty,\ i=1,2)$, driven by a

fractional Brownian motion of the form

$$\begin{cases} \frac{\partial}{\partial t}[x(t,\xi)+g(t,x(t-\tau_1,\xi))] = \frac{\partial^2}{\partial^2\xi}[x(t,\xi)+g(t,x(t-\tau_1,\xi))] \\ +\int_0^t b(t-s)\frac{\partial^2}{\partial^2\xi}[x(s,\xi)+g(s,x(s-\tau_1,\xi))]ds \\ +f(t,x(t-\tau_2,\xi))+\mu(t,\xi)+\sigma(t)\frac{dB^H}{dt}(t), \qquad t\geq 0, \\ x(t,0)+g(t,x(t-\tau_1,0))=0, \qquad t\geq 0, \\ x(t,\pi)+g(t,x(t-\tau_1,\pi))=0, \quad t\geq 0, \\ x(s,\xi)=\varphi(s,\xi), \;\; -\tau\leq s\leq 0 \quad a.s., \end{cases} \tag{4.1}$$

where $B^H(t)$ is a fractional Brownian motion, $f,\ g:\mathbb{R}^+\times\mathbb{R}\longrightarrow\mathbb{R}$ are continuous functions and $b:\mathbb{R}^+\longrightarrow\mathbb{R}$ is continuous function and $\varphi:[-\tau,0]\times[0,\pi]\longrightarrow\mathbb{R}$ is a given continuous function such that $\varphi(s,.)\in L^2([0,\pi])$ is measurable and satisfies $\mathbb{E}\|\varphi\|^2<\infty$.

We rewrite (4.1) into abstract form of (1.1), let $X=L^2([0,\pi])$. Define the operator $A:D(A)\subset X\longrightarrow X$ given by $A=\frac{\partial^2}{\partial^2\xi}$ with domain

$$D(A)=H^2([0,\pi])\cap H_0^1([0,\pi]),$$

then we get

$$Ax=\sum_{n=1}^{\infty}n^2<x,e_n>_X e_n, \quad x\in D(A),$$

where $e_n:=\sqrt{\frac{2}{\pi}}\sin nx,\ n=1,2,....$ is an orthogonal set of eigenvector of $-A$.

It is well known that A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $\{S(t)\}_{t\geq 0}$ in X, thus $(\mathcal{A}.1)$ is true. Furthermore, $\{S(t)\}_{t\geq 0}$ is given by (see [15])

$$S(t)x=\sum_{n=1}^{\infty}e^{-n^2t}<x,e_n>e_n$$

for $x\in X$ and $t\geq 0$, that satisfies $\|S(t)\|\leq e^{-\pi^2 t}$ for every $t\geq 0$.
Let $B:D(A)\subset X\longrightarrow X$ be the operator given by

$$B(t)z=b(t)Az \qquad \text{for } t\geq 0 \text{ and } z\in D(A).$$

We assume that the following conditions hold:

(i) Let $Hu:[0,T]\longrightarrow X$ be defined by

$$Hu(t)(\xi)=\mu(t,\xi),\ 0\leq\xi\leq\pi,\ u\in L^2([0,T],U).$$

(ii) Assume that the operator $W:L^2([0,T],U)\longrightarrow X$ given by

$$Wu(\xi)=\int_0^T R(T-s)\mu(t,\xi)ds,\ \ 0\leq\xi\leq\pi,$$

has a bounded invertible operator W^{-1} and satisfies condition $(\mathcal{H}.6)$. For the construction of the operator W and its inverse, see [16].

(iii) for $t\in[0,T]$, $f(t,0)=g(t,0)=0$,

(iv) there exist positive constants C_1, and C_3, $C_3<\frac{1}{2}$, such that

$$|f(t,\xi_1)-f(t,\xi_2)|\leq C_1|\xi_1-\xi_2|,\ \text{for } t\in[0,T] \text{ and } \xi_1,\xi_2\in\mathbb{R},$$

$$|g(t,\xi_1)-g(t,\xi_2)|\leq C_3|\xi_1-\xi_2|,\ \text{for } t\in[0,T] \text{ and } \xi_1,\xi_2\in\mathbb{R},$$

(v) there exist positive constants C_2 and C_4, such that

$$|f(t,\xi)| \leq C_2(1+|\xi|^2),\ \text{ for } t\in[0,T] \text{ and } \xi\in\mathbb{R},$$
$$|g(t,\xi)| \leq C_4(1+|\xi|^2),\ \text{ for } t\in[0,T] \text{ and } \xi\in\mathbb{R},$$

(vi) the function $\sigma : [0,+\infty) \to \mathcal{L}_2^0(L^2([0,\pi]), L^2([0,\pi]))$ satisfies

$$\int_0^T \|\sigma(s)\|^2_{\mathcal{L}_2^0} ds < \infty,\ \ \forall T>0.$$

Define the operators $F, G : \mathbb{R}^+ \times L^2([0,\pi]) \longrightarrow L^2([0,\pi])$ by

$$F(t,\phi)(\xi) = f(t,\phi(-\tau_1)(\xi)) \ \text{ for } \xi\in[0,\pi] \text{ and } \phi\in L^2([0,\pi]),$$

and

$$G(t,\phi)(\xi) = g(t,\phi(-\tau_2)(\xi)),\ \ \text{and } \phi\in L^2([0,\pi])$$

If we put

$$\begin{cases} x(t)(\zeta) = x(t,\zeta),\ t\in[0,T] \text{ and } \ \zeta\in[0,\pi] \\ x(t,\zeta) = \varphi(t,\zeta),\ t\in[-\tau,0] \text{ and } \ \zeta\in[0,\pi], \end{cases} \tag{4.2}$$

then, the problem (4.1) can be written in the abstract form

$$\begin{cases} d[x(t)+G(t,x(t-r(t)))] & = [Ax(t)+G(t,x(t-r(t)))]dt + \int_0^t B(t-s)[x(s) \\ & +G(s,x(s-r(s)))]dsdt + [F(t,x(t-\rho(t)))+Hu(t)]dt \\ & +\sigma(t)dB^H(t),\ 0\leq t\leq T, \\ x(t)=\varphi(t),\ -\tau\leq t\leq 0. & \end{cases}$$

Moreover, if b is bounded and C^1 function such that b' is bounded and uniformly continuous, then $(\mathcal{A}.1)$ and $(\mathcal{A}.2)$ are satisfied and hence, by Theorem 2.5, Equation (4.1) has a resolvent operator $(R(t))_{t\geq 0}$ on X. As a consequence of the continuity of f and g and assumption (iii) it follows that F and G are continuous. By assumption (iv), one can see that

$$\|F(t,\phi_1)-F(t,\phi_1)\|_{L^2([0,\pi])} \leq C_1\|\phi_1-\phi_2\|_{L^2([0,\pi])},$$
$$\|G(t,\phi_1)-G(t,\phi_1)\|_{L^2([0,\pi])} \leq C_3\|\phi_1-\phi_2\|_{L^2([0,\pi])},\ \ \text{with } \ C_3<\frac{1}{2}.$$

Furthermore, by assumption (v), it follows that

$$\|F(t,\phi)\| \leq C_2(1+\|\phi\|^2),\ \text{ for } t\in[0,T],$$
$$\|G(t,\phi)\| \leq C_4(1+\|\phi\|^2),\ \text{ for } t\in[0,T].$$

then all the assumptions of Theorem 3.3 are fulfilled. Therefore, we conclude that the system (4.1) is controllable on $[-\tau,T]$.

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